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TWELVE GENERAL POSITION POINTS ALWAYS FORM THREE INTERSECTING TETRAHEDRA

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This note answers affirmatively the following question which appeared in a list of problems [3] generated by a July, 1977 Discrete Geometry conference at Oberwolfach: "Given 12 points in general position in space, can one always put them into 3 sets of 4 each so that the 3 tetrahedra have an interior point in common?" Related conjectures are considered.

1. Introduction

A set S of at least $d+1$ points in $\mathbf{R}^d$ is said to be in *general position* provided no $d+1$ of them lie in any hyperplane of $\mathbf{R}^d$. This is a very weak type of independence; it does imply that at most 2 points can lie on any line, at most 3 in any 2-dimensional plane, at most 4 in any 3-dimensional subspace (or flat), and so on, up to dimension d. But it does not exclude pairs of points from forming lines which are a *pencil* (all the lines go through one point), nor does general position prevent triples of points from forming planes which are a "*book*" (all planes have a line in common). The following is a stronger type of independence which implies general position, and which has been useful in several contexts (see [1] and [4]). Here $|S|$, aff S, conv S, int S, and dim S will denote, respectively, the cardinality of S, the smallest flat (translate of a linear space) containing S, the convex hull of S, the (topological) interior of S, and the dimension of the affine hull aff S. (The empty set and a singleton set are considered to be of dimension -1, and 0, respectively.)

A set $S\subset\mathbf{R}^d$ is said to be *strongly independent* provided that each finite family $\{S_1,\ldots,S_r\}$ of pair wise disjoint subsets of S has the property: If $d_i=|S_i|-1\leq d$, then

$$\dim\left(\bigcap_{i=1}^{r}\operatorname{aff} S_i\right)=\max\left(-1,\, d-\sum_{i=1}^{r}(d-d_i)\right). \tag{1}$$

(Condition (1) may be thought of as follows: Since $(d-d_i)$ is just the deficiency of aff S_i when S_i is in general position, condition (1) implies the general position of S and its subsets. Thus the right side of the equation is essentially the dimension of the space reduced by the deficiencies of the flats aff S_i. This keeps the flats aff S_i from forming "pencils of lines", "books of planes", etc.)

An even stronger type of independence was used by Tverberg [6] in proving Theorem A below. A set S of m distinct points in $\mathbf{R}^d$ is said to be *algebraically independent* provided the md real coordinates of the m points in S are algebraically independent over the field of rational numbers. Although this algebraic independence plays an important role in the proof, it does not occur in the statement of Tverberg's theorem:

Theorem A (1966). *Each set S of at least $[(d+1)(r-1)+1]$ points in $\mathbf{R}^d$ has an r-partition $S=S_1\cup\cdots\cup S_r$ (into pair wise disjoint subsets) so that the intersection $\bigcap_{i=1}^r \operatorname{conv} S_i$ of the convex hulls is not empty.*

It was natural to ask how large the set $S\subset\mathbf{R}^d$ had to be to assure that the above intersection (in Tverberg's theorem) was not only non-empty, but k-dimensional. It is clear that if $k>1$ then some independence condition was necessary as well as a lower bound on the cardinality of S, since *any* number of points on a line could never form sets of dimension greater than 1. An answer was given in Reay [4]:

Theorem B (1968). *Each set S of at least $[(d+1)(r-1)+k+1]$ strongly independent points in $\mathbf{R}^d$, $0\leq k\leq d$, has an r-partition $S=S_1\cup\cdots\cup S_r$ so that $\dim(\bigcap_{i=1}^r \operatorname{conv} S_i)\geq k$.*

The proof of Theorem B made essential use of the hypothesis of strong independence of S. But in several special cases ($d=2$ or $r=2$) the strong independence of S could be replaced by the weaker condition that S was in general position. In particular, it is shown in [4] that any $3r$ points in general position in $\mathbf{R}^2$ may be partitioned to form r triangles with a common interior point (Corollary 6), and any $2(d+1)$ points in $\mathbf{R}^d$ in general position may be partitioned to form 2 simplices with a common interior point (Theorem 4). The problem stated in the above abstract is actually one of the first unsolved cases ($d=r=k=3$) of a more general conjecture, stated in [4]:

Conjecture 1. Theorem B remains true in general with strong independence replaced by general position.

It is unfortunate that the proof in the next section does not seem to generalize to higher values of d and r, but perhaps not surprising, since the more general conjecture has remained unsolved for a decade, and strong independence has been the property most frequently considered in various extensions. (See [1].)

2. Solution of the problem

Theorem. *Any set S of 12 points in general position in $\mathbf{R}^3$ has a 3-partition $S=S_1\cup S_2\cup S_3$ into sets whose convex hulls have a common interior point, that is,*

$$\bigcap_{i=1}^{3}(\operatorname{int}\operatorname{conv} S_i)\neq\emptyset.$$

We first establish several useful lemmas.

Lemma 1. *Let M be a 3×3 matrix of zeros and ones so that each row contains both a 0 and a 1. Then the rows of M may be interchanged so that the new diagonal of M is neither all zeros nor all ones.*

Proof. If the diagonal of M is $(1,1,1)$ then exchanging rows 2 and 3 puts a 0 on the diagonal unless rows 2 and 3 are both $(0,1,1)$. Then interchanging rows 1 and 2 gives both a 0 and 1 on the diagonal. The parallel argument when the diagonal of M is $(0,0,0)$ completes the proof.

Lemma 2. *Let $P=\{\alpha_1,\alpha_2,\alpha_3\}$ be a planar pencil of 3 lines and $T=\{x_1,x_2,x_3\}$ be a set of 3 distinct points of the plane not on the pencil P. Then there is a 1–1 correspondence between P and T (which we will assume to be $\alpha_i \leftrightarrow x_i$, $i=1,2,3$) with this property: If H_i denotes the closed half-space which contains x_i and has boundary α_i, then the intersection $\bigcap_{i=1}^{3}(\operatorname{int} H_i)$ is non-empty.*

Proof. The pencil P of 3 lines is shown in Fig. 1. One side of each line is designated by an arrow and denotes the "1-side" of the line, while the other side will be denoted the "0-side". In this way each of the six planar regions formed by the pencil has a unique representation as a triple of zeros and ones; the ith coordinate denoting on which side of α_i the region lies , $i=1,2,3$.

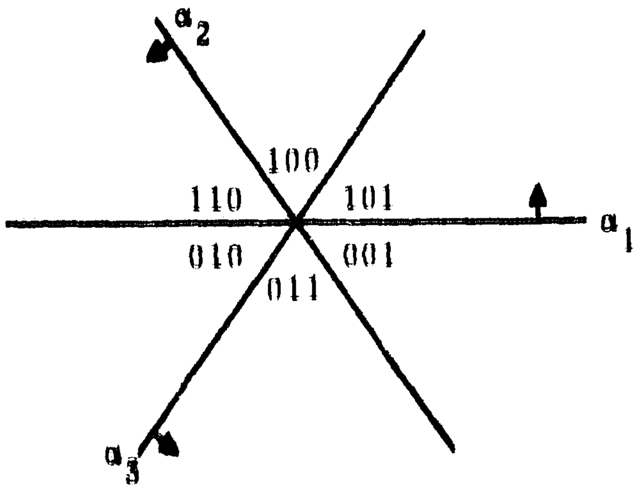

Fig. 1.

Let M denote the 3×3 zero-one matrix whose ith row ($i=1,2,3$) is the representation of the region in which the point x_i lies. By Lemma 1 we may assume the diagonal of M represents one of the 6 regions. Thus the correspondence $\alpha_i \leftrightarrow x_i$ produces the three desired half-planes H_i whose intersection is the region represented by the diagonal of M. Hence the interior of $\bigcap_{i=1}^{3} H_i$ has points arbitrarily near the center of the pencil, and in particular, is non-empty.

We now consider the main result.

Proof of the theorem

Note that the theorem is sharp in the sense that at least 12 points are necessary (4 for each of the sets S_i), since the convex hull of any 3 or fewer points in $\mathbf{R}^3$ has empty interior.

Let S by any set of 12 points in general position in $\mathbf{R}^3$ (so no 4 points lie in any plane). Theorem A with $d=r=3$ asserts that there are three pair wise disjoint subsets S_1, S_2, S_3 of S with $\bigcap_{i=1}^3 \operatorname{conv} S_i \neq \emptyset$ and $a+b+c \leq 9$ where a, b and c denote, respectively, the cardinality of S_1, S_2, and S_3. After translating S and possibly removing some unnecessary points from each S_i, it is no loss of generality to assume that $0 \in \bigcap_{i=1}^3 \operatorname{rel\,int\,conv} S_i$ and $a \leq b \leq c$. (Here rel int X denotes the interior of set X relative to the smallest flat which contains X. If $X=\{x\}$ is a single point, then rel int $X=\{x\}$ by definition.)

Case 1. If $a=1$ so that $S_1=\{0\}$ is a single point, then S_2 and S_3 must each be the vertex set of a tetrahedron whose interior contains 0, for otherwise some 4 points of S lie in a plane and general position is denied. Thus all points sufficiently near 0 lie in $\bigcap_{i=2}^3 \operatorname{int\,conv} S_i$. Now let S_1 be enlarged by the 3 extra points of S so that conv S_1 is also now a tetrahedron with interior points arbitrarily near the vertex 0. Clearly the three tetrahedra have interior points in common, and the proof is complete in this case. Thus assume $a \geq 2$.

Case 2. If $b=2$ then the four points in $S_1 \cup S_2$ determine two intersecting segments, so they are coplanar. Hence general position is denied. Thus assume $b \geq 3$.

Case 3. If $c=4$, then 0 lies in the interior of the tetrahedron conv S_3, and $a=2$ and $b=3$ since $a+b+c \leq 9$. From general position we conclude that the line segment $\sigma_1=\operatorname{conv} S_1$ must meet the (relative 2-dimensional) interior of the triangle $\sigma_2=\operatorname{conv} S_2$. See Fig. 2. When S_2 is enlarged by adding one of the extra points, the tetrahedron thus formed will have σ_2 as one of its faces, and line segment c_1 enters the interior of this tetrahedron through face σ_2 at 0. Enlarging S_1 by the two remaining extra points produces a tetrahedron with interior points arbitrarily near each point of σ_1. Hence the three tetrahedra have common interior points near 0. Thus assume $2 \leq a \leq 3 = b = c$.

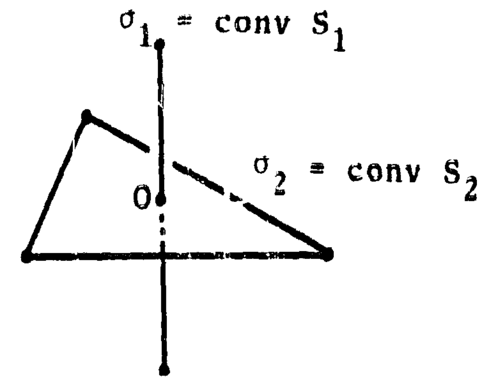

Fig. 2.

With these assumptions S_2 and S_3 determine 2-dimensional subspaces of $\mathbf{R}^3$, with a line L in common. See Fig. 3. Since $0 \in \operatorname{rel\,int\,conv} S_i$, a neighborhood of 0 along L lies in $\bigcap_{i=2}^3 \operatorname{conv} S_i$. Further, either (case 4) S_1 determines a triangle

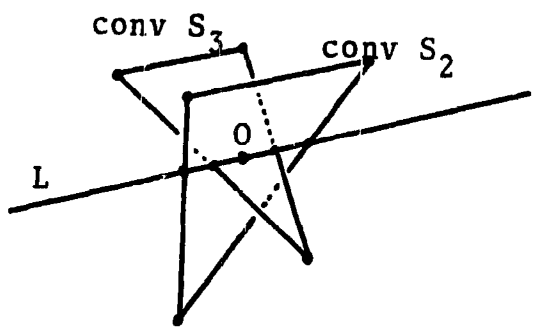

Fig. 3.

which meets L only at the interior point 0, or (case 5) S_1 determines a triangle which meets L in a neighborhood of 0, in which case the 3 triangles form a "book of planes" with common line L, or else (case 6) S_1 determines a line segment which could meet L only at 0. Then $a = 3$ in cases 4 and 5 while $a = 2$ in case 6.

Case 4. In this case the 3 triangles conv S_i each contain 0 and determine 3 planes H_i with only 0 in common. Assign one of the 3 remaining points of S to each set S_i. Each expanded set S_i' determines a tetrahedron and all points sufficiently near 0, lying on one side of the plane H_i, must lie in int conv S_i. Since the intersection of 3 independent half-spaces in $\mathbf{R}^3$ must have one octant in common, it follows that points sufficiently near 0 lie in $\bigcap_{i=1}^3$ int conv S_i' and the proof is complete in this case.

To handle the two remaining cases, let $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ be the projection onto $L^\perp$ with kernel L. Then the sets πS_i determine a pencil of 3 line segments through 0 in this plane. See Fig. 4. Let $\{\pi x_1, \pi x_2, \pi x_3\}$ be the projection of three of the remaining points of S onto the plane $L^\perp$ which do not lie on the pencil. By Lemma 2 we may assume that the 3 half-planes H_i formed by the πS_i and πx_i have common interior points arbitrarily near zero.

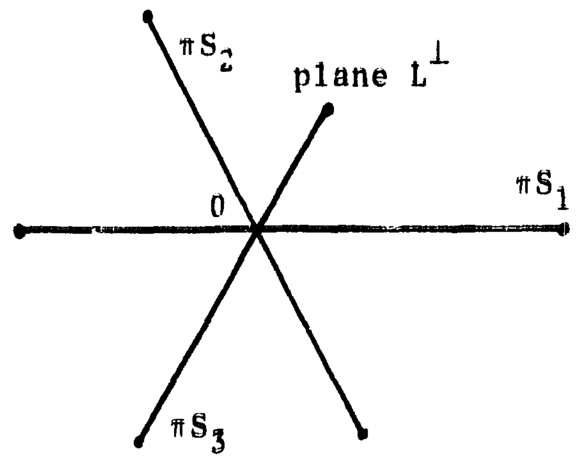

Fig. 4.

Case 5. Each set $S_i' = S_i \cup \{x_i\}$ is the vertex set of a tetrahedron. The point 0 lies in the relative interior of a triangular face of this tetrahedron conv S_i' and all of its points project onto $H_i \subset L^\perp$. Thus the three tetrahedra have common interior points.

Case 6. Reasoning here parallels case 5 except that $S \sim (\bigcup S_i) = \{x_1, x_2, x_3, y\}$. Let $S_i' = \{x_i\}$ for $i = 2, 3$ and let $S_1' = S_1 \cup \{x_1, y\}$. Then 0 lies in the relative interior of an edge of $\operatorname{conv} S_1'$ and in the relative interior of a face of the other 2 tetrahedra. As above there is a point near 0 which projects into each H_i and which lies interior to each of the three tetrahedra $\operatorname{conv} S_i$. This completes the proof.

3. Related results, open problems and conjectures

The methods in the proof of the last section may be extended to establish the following two results.

Proposition A. *Any set S of 11 points in general position in $\mathbf{R}^3$ has a 3-partition $S = S_1 \cup S_2 \cup S_3$ into sets whose convex hulls have a triangle in common, that is,*

$$\dim\left(\bigcap_{i=1}^{3} \operatorname{conv} S_i\right) = 2.$$

This is one additional case ($d = r = 3$, $k = 2$) of Conjecture 1 stated in Section 1 above. Note that some sort of independence like general position is clearly still necessary. Of more interest is last remaining case when $d = r = 3$, namely $k = 1$, since hopefully the general position hypothesis could be weakened:

Proposition B. *Any set S of 10 points in general position $\mathbf{R}^3$ has a 3-partition $S = S_1 \cup S_2 \cup S_3$ into sets whose convex hulls have a line segment in common, that is,*

$$\dim\left(\bigcap_{i=1}^{3} \operatorname{conv} S_i\right) = 1.$$

The known proofs of these results use brute force rather than elegance, and are omitted. Hopefully short methods of proof will be found which might apply to the open cases $d \geq 3$ and $r \geq 3$ of the general conjecture.

The following example from [5] shows that Proposition B does not remain valid without the general position hypothesis unless the cardinality of S is increased from 10 to at least $14 = 2d(r-1)+2$.

Example. Let S be an $(r-1)$*-fold cross basis* in $\mathbf{R}^d$, that is, $S = \{\alpha b \mid b \in B$ and $\alpha = 0, \pm 1, \pm 2, \ldots, \pm(r-1)\}$ where B is any linear basis for $\mathbf{R}^d$. Then $|S| = 2d(r-1)+1$ and the origin is the only possible *r-divisible point of S*, that is, the only point p for which there exists an r-partition $S = S_1 \cup \cdots \cup S_r$ with $p \in \bigcap_{i=1}^{r} \operatorname{conv} S_i$. This is easy to see from the fact that each point $p \neq 0$ is contained in a closed half-space H with $|S \cap H| \leq r-1$, so p could not be an r-divisible point of S. The set S is certainly not in general position. This example shows that the lower bound on S in the following is the best possible.

Conjecture 2. Any set S of at least $2d(r-1)+2$ in $\mathbf{R}^d$ has an r-partition $S = S_1 \cup \cdots \cup S_r$ into sets whose convex hulls have a line segment in common, that is,

$$\dim\left(\bigcap_{i=1}^{r} \operatorname{conv} S_i\right) = 1.$$

Eckhoff [2] has established this conjecture in the special case $r = 2$, and Reay [5] has shown that it holds when $d = 2$ or when the set of r-divisible points of S is convex. The following result from [5] adds further strength to Conjecture 2, and the bound is sharp.

Proposition C. *Any set S of $2d(r-1)+2$ points in $\mathbf{R}^d$ admits two distinct r-divisible points.*

Note added in proof

In an article "Radon partitions with k-dimensional intersection" to appear in the Journal of the London Mathematical Society, J.-P. Doignon has recently shown that case $r = 2$ of Conjecture 1 remains true under conditions weaker than general position.

References

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